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Hugh MacColl after One Hundred Years

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MacColl's Modes of Modalities

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Résumé : Hugh MacColl est présenté d'ordinaire comme un pionnier des logiques modales et multivalentes, suite à son introduction de modalités qui vont au-delà de la simple vérité et fausseté. Mais un examen plus attentif montre que cet héritage est discutable et devrait tenir compte de la façon dont ces modalités procédaient. Bien que MacColl ait conçu une logique modale au sens large du terme, nous montrerons qu'il n'a pas produit une logique multivalente au sens strict. Sa logique serait comparable plutôt à une « logique non-fregéenne », c'est-à-dire une logique algébrique qui effectue une partition au sein de la classe des vérités et faussetés mais n'étend pas pour autant le domaine des valeurs de vérité.

Abstract: Hugh MacColl is commonly seen as a pioneer of modal and many-valued logic, given his introduction of modalities that go beyond plain truth and falsehood. But a closer examination shows that such a legacy is debatable and should take into account the way in which these modalities proceeded. We argue that, while MacColl devised a modal logic in the broad sense of the word, he did not give rise to a many-valued logic in the strict sense. Rather, his logic is similar to a "non-Fregean logic": an algebraic logic that partitions the semantic classes of truth and falsehood into subclasses but does not extend the range of truth-values.

Modalities and many-valuedness

A preliminary attention to the notation is in order. MacColl's Logic (hereafter: MCL) resorts to a symbolic language that is in accordance with the algebraic style of George Boole or Ernst Schröder. Mathematical signs are used to characterize operations between any *propositions* A^B and C^D. Thus, $A^B + C^D$ stands for their sum, A^BC^D (or $A^B \times C^D$, or $A^B.C^D$) for their product, $A^B : C^D$ for the implication from A^B to C^D, $A^B = C^D$ for their equivalence

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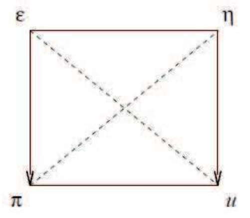
(synonymous with $(A^B : C^D)(C^D : A^B)$), and $(A^B)'$ for the denial of A^B. MCL is thus a logic of predication that consists in propositions subsuming a class under another one: in A^B, the first term A is the subject term while the second term B is the predicate term whose application results in a proposition such as "A is B", or "The A is a B".

A first terminological distinction is that between the occurrence of propositions as terms or factors and as exponents. A^B and C^D occur as *terms* in the sum $A^B + C^D$ and as *factors* in the product $A^B C^D$. Also, A and C are the subjects of the predications A^B and C^D; while B and D occur as *exponents*, because they are predicates to which A and C respectively belong. Another distinction is that between the adjectival and predicative use: B is used *adjectivally* in A_B, and *predicatively* in A^B, so that the whole proposition A_B^C means that the A which is a B is also a C.

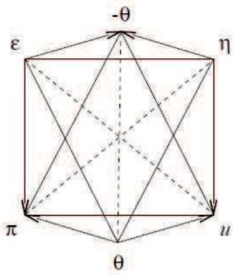
On the one hand, this symbolic device helps to convey information about *quantity* by introducing integers into the adjectival and predicative uses. Thus $A_1, A_2 \dots, A_n$ specify n different individuals in the class A, and the superscript 0 gives an expression to *quantification* in terms of the null class: A_B^0 means that no A is B ("the A that is B belongs to the null class"), and $A_B^{\text{-}0}$ that some A is B ("the A that is B does not belong to the null class"); $A_{\text{-}B}^0$ means that every A is B ("the A that is not B belongs to the null class"), and $A_{\text{-}B}^{\text{-}0}$ that some A is not B ("the A that is not B does not belong to the null class"). On the other hand, the later distinction between metalanguage and object-language doesn't appear in MCL: the truth-values are not confined to a separate higher-order language and are treated on a par with any other term of the symbolic logic. The result is a class of semantic predicates that go beyond truth and falsehood, as stated by MacColl:

> The symbol A^τ only asserts that A is true in a particular case or instance. The symbol A^ε asserts more than this: it asserts that A is *certain*, that A is *always* true (or true in *every case*) within the limits of our data and definitions, that its probability is 1. The symbol A^ι only asserts that A is false in a particular case or instance; it says nothing as to the truth or falsehood of A in other instances. The symbol A^η asserts more than this; it asserts that A contradicts some datum or definition, that its probability is 0. Thus A^τ and A^ι are simply *assertive*; each refers to one case, and raises no general question as to data or probability. The symbol A^θ (A is a *variable*) is equivalent to $A^{\text{-}\eta} A^{\text{-}\varepsilon}$; it asserts that A is neither *impossible* nor *certain*, that is, that A is *possible* but *uncertain*. In other words, A^θ asserts that the probability of A is neither 0 nor 1, but some proper fraction between the two. [MacColl 1906, 7]

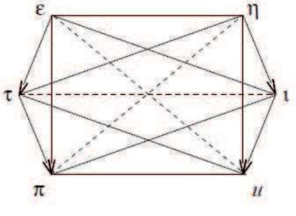
MacColl supplements the five preceding semantic predicates with four other elements, *i.e.* π for possibility, p for probability, q for improbability, and u for uncertainty [MacColl 1906, 14]. Most of these predicates (except for p and q) behave like modalities, since they are reminiscent of the Aristotelian logic and its isomorphic oppositions within quantified and modal propositions: certainty ("every A is B") and impossibility ("every A is not B") correspond to universal affirmation and negation, while possibility ("some A is B") and uncertainty ("some A is not B") correspond to particular affirmation and negation. As to the remaining cases of truth, falsehood, variability, probability and improbability, they require an extension of the Aristotelian square: variability ("some A is B , some A is not B") corresponds to two-sided possibility or contingency, and such a concept finds its rightful place in the first logical hexagon of Robert Blanché [Blanché 1953]; truth ("this A is B") and falsehood ("this A is not B") correspond not to universal or particular, but *singular* propositions, whose rightful place requires further extension from the second logical hexagon of Tadeusz Czeżowski [Czeżowski 1955] to the logical octagon of Jan Woleński [Woleński 1998] (see figures [1–4]).



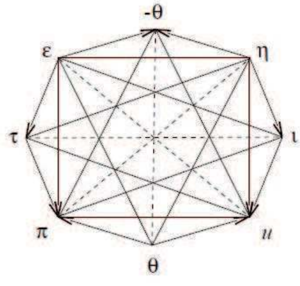
[Fig. 1]: Aristotle's Square



[Fig. 2]: Blanché's Hexagon



[Fig. 3]: Czeżowski's Hexagon



[Fig. 4]: Woleński's Octagon

The result is the following list of 28 logical oppositions and their five kinds of opposition: [1]

Contraries (5): $\{\varepsilon, \eta\}; \{\varepsilon, \theta\}; \{\eta, \theta\}; \{\eta, \tau\}; \{\varepsilon, \iota\}$

Subcontraries (5): $\{\pi, u\}; \{\pi, \text{-}\theta\}; \{u, \text{-}\theta\}; \{\pi, \iota\}; \{u, \tau\}$

Contradictories (4): $\{\varepsilon, u\}; \{\eta, \pi\}; \{\text{-}\theta, \theta\}; \{\tau, \iota\}$

Subalterns (10): $\{\varepsilon, \pi\}; \{\eta, u\}; \{\theta, \pi\}; \{\theta, u\}; \{\varepsilon, \text{-}\theta\}; \{\eta, \text{-}\theta\}; \{\varepsilon, \tau\}; \{\eta, \iota\}; \{\tau, \pi\}; \{\iota, u\}$

Mere non-contradictories (4): $\{\text{-}\theta, \iota\}; \{\text{-}\theta, \tau\}; \{\iota, \theta\}; \{\tau, \theta\}$

The 28 oppositions above exhaust the set of logical relations between MacColl's modalities, excluding the peculiar cases of p and q: the latter are probable values that cannot be located within a polygon of oppositions, given that they represent any of the indefinitely many values between full truth (or certainty: 1) and full falsehood (or impossibility: 0). The dotted lines stand for contradictory relations; the arrows indicate the entailment relation between an antecedent A and its consequent B, such that $A : B$ is a theorem. Here is a sample of those theorems from MacColl [MacColl 1906, 8]: (T12) $(A^\tau + A^\iota)$ expresses the law of excluded middle between contradictory terms, (T14) $(A^\varepsilon + A^\eta + A^\theta)^\varepsilon$ means that three modalities exhaust the semantic class of MCL, while the implications (T15) $A^\varepsilon : A^\tau$ and (T16) $A^\eta : A^\iota$ correspond to subalternation relations.

1 and 0 are usually used to symbolize truth and falsehood, in algebraic semantics; but it has been frequently claimed by the commentators on MacColl (including [Simons 1998]) that he thought of truth-values in probabilistic terms of truth-cases. If so, then any of the logical values of MCL should be reformulated within the closed interval [0,1] of an infinitely-valued logic. This should not be a proper characterization of MacColl's view of modalities, however: the difference between particular and singular judgments cannot be rendered within such a probabilistic line, and the above oppositions depict logical relations that are qualitative rather than quantitative (as witnessed by the absence of p and q from the polygons).

At any rate, it may seem strange to state logical oppositions between concepts expressing truth-values: if an opposition is about the possibility for two terms to be both true or false, what does it mean for τ and ι not to be true or false together? A plausible answer is that these alleged truth-values are not that but, rather, modal operators attached to propositions. Hence the ensuing problem is to specify the logical status of the elements that compose the semantic class. In other words: is MCL a *modal* logic, or a *many-valued* logic?

1. For every polygon of opposition with n vertices, the number of logical oppositions is $(n - 1) + ((n - 1) - 1) + \ldots + 1$. Thus a square has $n = 4$ vertices and $(4 - 1) + (3 - 1) + 1 = 6$ logical oppositions. We do not count four but five kinds of opposition in the above octagon: the so-called "mere non-contradictories" are neither a case of subalternation nor a case of subcontrariety, as witnessed by the pair $\{\theta, \tau\}$ which does not express any entailment relation between its terms. See [Smessaert 2009] about the definitional link between subalternation and entailment.

To begin with, only five elements are brought out in the semantic class because only these are independent and cannot be reduced to each other; conversely, possibility is the negation of impossibility ($\pi = -\eta$) and uncertainty is the negation of certainty ($u = -\varepsilon$). For this reason, we restrict our attention to the nature of the five elements of the semantic class *Five* = $\{\varepsilon, \eta, \tau, \iota, \theta\}$.

Let A be a proposition of a language and v a valuation function that maps A onto an element x of *Five*, so that $v(\mathrm{A}) = x$. We are then faced with two alternatives. Either MCL is a modal logic, in which case the semantic predicates are not elements of *Five* but unary operators Ⓒ that are attached to propositions and form modal propositions of the form $A =$ Ⓒp. Or MCL is a many-valued logic, and the semantic predicates are not operators but the resulting values x of an operation. Or can they be both, in one and the same symbolic logic? There have been such cases after MacColl's death, in the realm of many-valued logics: in Jan Łukasiewicz's three-valued system Ł$_3$, the modality of possibility occurred both as a modal operator Ⓒ $=$ M and a third value $x = 1/2$ [Łukasiewicz 1920]; the same obtains in Dmitri Bochvar's three-valued logic of assertion, where truth and falsehood occur both as truth-values and unary operators meaning "It is true that" or "It is false that" [Bochvar 1938].

Whatever the case may be, it can be said more generally that MCL is a logic of modalities, without specifying the mode in which these modalities are introduced within the formal system. The five semantic predicates occur as terms, exponents or factors, but never in the specific manner of truth-values or unary operators: these characterizations belong to Gottlob Frege's modern or functional logic, and MacColl never adhered to this logical turn as his most famous opponent Bertrand Russell did.

Despite this theoretical discrepancy, MCL is closely related to Bochvar's three-valued logic.

Firstly, MacColl and Bochvar insisted equally upon the connection between *affirmation* and *truth*, *denial* and *falsehood*. MacColl claims that any proposition A is equivalent with its being true, so that $A = A^{\tau}$. Bochvar explicitly renders this by attaching an operator of affirmation * to A, so that A^* means the same as "It is true that A". But just as not every occurrence of the proposition A entails its being true, MacColl argued that A and A^{τ} are equivalent without being synonymous with each other:

> It may seem paradoxical to say that the proposition A is not quite synonymous with A^{τ}, nor A' with A^{ι}; yet such is the fact. Let A = *It rains.* Then A' = *It does not rain*; A^{τ} = *it is true that it rains*; and A^{ι} = *it is false that it rains.* The two propositions A and A^{τ} are *equivalent* in the sense that each *implies* the other; but they are not *synonymous*, for we cannot always substitute the one for the other. In other words, the equivalence $(A = A^{\tau})$ does not necessarily imply the equivalence $\phi(A) = \phi(A^{\tau})$. [MacColl 1906, 16]

This is characteristic of what has been repeatedly blamed in MacColl's logic, namely: the equivocal use of its symbols. For A and A^τ are equivalent when the term A occurs as a *statement* (an interpreted function) uttered by a speaker, while A and A^τ are not synonymous when the term A occurs as a *propositional function* that is not stated and unrestrictedly ranges over the elements of *Five*. For if A is uttered, then A is claimed to be true by its speaker and every such proposition is trivially certain: if $A = \tau$ then $A^\tau = \tau^\tau = \varepsilon$, and if $A = \varepsilon$ then $A^\tau = \varepsilon^\tau = \varepsilon$. Now if A' is uttered, then A is claimed to be false by its speaker and every such proposition is trivially impossible. At the same time, if A is not stated and occurs as an unasserted form of words (proposition) then A can range over any element of *Five*. Let $A = \theta_\tau$ and $\phi(A) = A^\varepsilon$; hence $\phi(A) = A^\varepsilon = \theta_\tau^\varepsilon = \eta$, while $\phi(A^\tau) = (A^\tau)^\varepsilon = (\theta_\tau^\tau)^\varepsilon = (\varepsilon)^\varepsilon = \varepsilon$. Therefore, $\phi(A) \neq \phi(A^\tau)$ for some values of A. Likewise, let $A = \theta_\iota$; hence $\phi(A') = (A')^\varepsilon = A^\eta = \theta_\iota^\eta = \eta$, while $\phi(A^\iota) = (A^\iota)^\varepsilon = (\theta_\iota^\iota)^\varepsilon = (\varepsilon)^\varepsilon = \varepsilon$. Therefore, $\phi(A') \neq \phi(A^\iota)$ for some values of A. Unlike Bochvar, MacColl thus assumed an implicit occurrence of the statements A and A' as truth- and falsehood-claims; whereas the fact that A and A^τ (or A' and A^ι) are non-synonymous betrays their explicit occurrence as propositional functions.

Secondly, MacColl foreshadowed Bochvar's subsequent distinction between *external* and *internal* negation. Such a terminology refers to the scope of functions, unlike MacColl's logic. But it makes sense in MCL, however: either its larger scope applies to the whole proposition A^B, and negation is thus said to be external; or its narrower scope applies to the subject term A only, and negation is thus said to be internal. This difference has already been exemplified in the preceding lines: in the upper vertex of the logical oppositions, $\text{-}\theta$ in $A^{\text{-}\theta}$ depicts an external negation meaning that the proposition A is *not* contingent, while $(A')^\theta$ means that the denial of A is contingent. MacColl motivated this distinction with reference to the higher degree statements:

> The symbol A^{BC} means $(A^B)^C$; it asserts that the statement A^B belongs to the class C, in which C may denote *true*, or *false*, or *possible*, &c. Similarly A^{BCD} means $(A^{BC})^D$, and so on. From this definition it is evident that $A^{\eta\iota}$ is not necessarily or generally equivalent to $A^{\iota\eta}$, nor $A^{\varepsilon\iota}$ equivalent to $A^{\iota\varepsilon}$. [MacColl 1906, 7]

In an external negation (EN) like $A^{\text{-}\eta}$, the denial applies to the factor (or predicate) and is symbolized by a hyphen; in an internal negation (IN) like $(A')^\eta$, the denial applies to the term A and is symbolized by a single inverted comma. The logical difference between both is obviously seen with the following equivalences, where the place of negation does crucially matter:

(IN) $A^{\iota\eta} = (A^\iota)^\eta = (A')^\eta = A^\varepsilon; A^{\iota\varepsilon} = (A^\iota)^\varepsilon = (A')^\varepsilon = A^\eta; A^{\iota\theta} = (A')^\theta = A^\theta$
(EN) $A^{\eta\iota} = (A^\eta)^\iota = (A^\eta)' = A^\varepsilon + A^\theta; A^{\varepsilon\iota} = (A^\varepsilon)^\iota = (A^\varepsilon)' = A^\eta + A^\theta; A^{\theta\iota} = (A^\theta)^\iota = (A^\theta)' = A^\varepsilon + A^\eta$

MacColl will relevantly call for internal negation in order to establish a good number of theorems in MCL, assuming the definition of implication in terms of denial and impossibility: $(A : B) = (AB')^\eta$. He argues for this by relying upon the usual meaning of denial:

> By the "denial of a certainty" is not meant $(A^\varepsilon)'$, or its synonym $A^{\text{-}\varepsilon}$, which denies that a particular statement A is certain, but $(A_\varepsilon)'$ or its synonym A'_ε, the denial of the *admittedly certain* statement A_ε. This statement A_ε (since a suffix or subscriptum is adjectival and not predicative) *assumes* A to be certain; for both A_x and its denial *assume* the truth of A_x. Similarly, the "denial of a possibility" does not mean $A^{\text{-}\pi}$ but A'_π, or its synonym $(A^\pi)'$, the denial of the *admittedly possible* statement A_π. [MacColl 1906, 15]

A modern way to put this point is to say that denial doesn't apply to a modal operator ($\Box$ for necessity, say) but to its propositional content: the internal negation of $\Box A$ is $\Box{\sim}A$, rather than ${\sim}\Box A$. But again, any comparison between MCL, modal logic and many-valued logic should be made with caution. Against the view that a logical system could be equally taken to be modal or many-valued (as Łukasiewicz did), it has been proved by James Dugundji [Dugundji 1940] that:

> There exists no finitely many-valued logic that is characteristic of any of the Lewis systems S1 to S5, because any finitely many-valued logic will contain tautologies that are not theorems of S5 (and a fortiori not of S1 to S4 either). [Rescher 1969, 192]

This means that either MCL is a (finitely) many-valued logic, so that it is a weaker system than the Lewis systems S1-S5; or MCL is a modal logic that can be characterized by a modal system between S1 and S5, in which case it is not a many-valued logic. Although MCL is variously presented as a modal and (finitely) many-valued logic, it cannot be both in the light of Dugundji's proof. So which one should it be?

At any rate, the preceding entails that MCL may be seen as a modal logic only in the broad, informal sense of a logic for modalities: a modality is viewed to be a mode of being true or false, irrespective of how it is regimented in a logical system, and MacColl follows this line by arguing that certainty or impossibility are the same as being true or false in every case. Let us consider first the modal features that are commonly associated with MacColl's system.

A modal logic?

Non-truth-functionality is usually taken to be an essential criterion of a modal logic: not every value of such a composed proposition is to be determined by the value of its components, as the case is with $\Box A$ with respect to its component A. Although such a criterion doesn't hold with some many-valued translations of modal logics, we have seen with Dugundji's proof that no modal system between S1 and S5 can be characterized by a many-valued matrix. A truth-functional many-valued logic could be appropriate for modal systems weaker than S1, consequently. At any rate, modal logic is not merely specified by a set of logical theorems. For instance, Georg Henrik von Wright proposed a classification that departs from MacColl's view of modalities. Thus he writes:

> We shall distinguish between truth-concepts or truth-categories and modal concepts or modal categories. The logic of truth-concepts we shall call *truth-logic*, and the logic of modal concepts we shall call *modal logic*. The basic truth-categories are the two so-called *truth-values*, viz. *truth* and *falsehood*. Further examples of truth-categories are the concept of a truth-function and the instances of such functions: negation, conjunction, disjunction, (material) implication, (material) equivalence, *tautology*, and *contradiction*. It is of some importance to observe that the words tautology and contradiction are used in this essay as names of truth-functions exclusively. The words in question are sometimes used as *synonyms* for certain modal words. [von Wright 1951, 1]

Given this distinction, MCL appears as a conflation of truth- and modal concepts, and MacColl *is* one of these logicians that would use tautology and contradiction as synonyms for the modalities of certainty and impossibility. The meaning of modal logic is not definite, according to him, but a question of choice for different sakes. Thus we find in MacColl:

> In the traditional logic any proposition A^B of the *first degree* is called a *pure* proposition, while any of my propositions A^{BC} or A^{BCD}, &c., of a *higher degree* would generally be considered a *modal proposition*; but upon this point we cannot speak with certainty, as logicians are not agreed as to the meaning of the word 'modal'. For example, let the pure proposition A^B assert that "*Alfred* will go to *Belgium*"; then $A^{B\varepsilon}$ might be read "*Alfred* will *certainly* go to *Belgium*", which would be called a *modal* proposition. Again, the proposition A^{-B}, which asserts that "*Alfred* will *not* go to *Belgium*", would be called a *pure* proposition; whereas $A^{B\iota}$, or its synonym $(A^B)^\iota$, which asserts that A^B is false, would, by most logicians, be considered a *modal* proposition. [MacColl 1906, 94]

Furthermore, truth and falsehood could be defined as 'null' modalities $\bigcirc$ similar to the redundant operator of affirmation in the Fregean logic. Thus a modern translation of A^τ and A^ι would result in the modal formulas $\bigcirc A$ and $\bigcirc{\sim}A$. Once this assumption is accepted, it remains to see which sort of modal logic MCL is.

Various answers have been given to this respect: Storrs McCall [McCall 1967] identifies MCL with S3 and Shahid Rahman [Rahman 1997] argues for S2 or S3. Stephen Read [Read 1998] claims that the stronger system T (developed after Clarence Irving Lewis by Robert Feys and von Wright) is characteristic of MCL – given the theorems (T15) $A^{\varepsilon} : A^{\tau}$ and (T16) $A^{\eta} : A^{\iota}$– while discarding any even stronger system like S4 or S5. Against McCall, Read argues that:

> [T]he characteristic axiom of S3 does not figure in the nine theses McCall attributes to MacColl—indeed, if it did, then since MacColl's logic is normal, as shown above (*i.e.* $\eta^{\pi} = \eta$), there would ensue reduction theses such as $a^{\pi\pi} = a^{\pi}$, characteristic of S4, since S4 is the union of S3 and T. Since MacColl explicitly endorses normality and denies any reduction laws, his logic is T. [Read 1998, 74]

Whatever the case may be, the initial question of a characteristic logical system for MCL does not happen to make good sense from a MacCollian point of view.

For one thing, a modal system is said to be stronger (or weaker) than another whenever the former set of logical theorems includes (or is included in) the latter. But such an inclusion relation can be established only for modal systems and interpreted within a Kripkean structure, where inclusion concerns the relations of accessibility (reflexivity, transitivity, and the like) between possible worlds. Following Dugundji's result, again, MCL cannot be seen with the modern glasses of modal logic and its model structures because it proceeds as a *finite* many-valued logic.

Moreover, Rahman [Rahman 1997] also claimed that MacColl advocated a logical pluralism such that no *unique* logical system should be characteristic of MCL. In this sense, the latter is not so much a logical system in the modern sense of the word, *i.e.* a closed set of theorems; rather, it is a symbolic language whose theorems depend upon the applied context of discourse. According to Rahman, Aristotle's modal syllogistic is akin to connexive logic and closely related to the system S2; but other modal discourses are conceivable and should require alternative sets of modal theorems.

Finally, MCL is all the more distinct from modern modal logics in that their *iteration* and reduction theses don't proceed in the same way. Thus a crucial difference is to be made between iterated modalities and higher degrees statements. The reason is that MacColl's modalities occur as inclusion relations, while the modern modalities stand for quantifiers. For any sequence of modalities with n modal terms, the n^{th} and ultimate (right-sided) modality occurs as a predicate while the $n - 1^{th}$ preceding modalities are parts and parcels of the predicated subject. Given the inclusion relation that holds from left to right between any sequence of exponents, the law of reduction from every such higher-degree to a first-order statement requires one to know whether the primary (left-sided) predicate is included into its (right-sided) successor.

Let us take the statement $A^{\eta\iota\varepsilon}$ as an example. According to MacColl:

> The symbol $A^{\eta\iota\varepsilon}$ may be read "It is certain that it is false that it is impossible that A is true"; which may be abbreviated into "A is certainly possible". [MacColl 1900, 75]

As a third degree statement, it amounts to $(A^{\eta\iota})^{\varepsilon}$ and means that it is certain that it is false that it is impossible that A. Given that $A^{\eta\iota} = (A^{\varepsilon} + A^{\theta})$ (see [MacColl 1901, 140]), it follows that $A^{\eta\iota\varepsilon} = (A^{\varepsilon} + A^{\theta})^{\varepsilon}$ and a modern translation of this statement matches with the preceding result. Using $\Diamond$ for possibility, $A^{\eta\iota\varepsilon}$ would be rendered by $\Box{\sim}({\sim}\Diamond A)$, *i.e.* $\Box\Diamond A$. $(A^{\varepsilon} + A^{\theta})^{\varepsilon}$ and $\Box\Diamond A$ hold equally, since they both mean that A is certainly possible. However, the reduction laws of iterated modalities in the modern modal systems don't hold in MCL: $\Box A \leftrightarrow \Box\Box A$ in $S4$, whereas $A^{\varepsilon} \neq A^{\varepsilon\varepsilon}$; and $\Box\Diamond A \leftrightarrow \Diamond A$ in $S5$, whereas $A^{\varepsilon\pi} \neq A^{\varepsilon}$.

Consequently, MCL can be viewed as a logic of modalities in the broad sense of containing modal expressions; but not as a logical system including modal operators, given the very different nature of its operations upon modalities as predicates. The set-theoretical aspect of these operations is connected with an algebraic reading of modalities; and since algebraic operations $(+, \times, :, =)$ apply to values including a *supremum* 1 and an *infimum* 0, we should be naturally led to conclude that MCL is a many-valued logic whereby modalities stand for a set of truth-values. Let us scrutinize this widespread opinion among the commentators.

A many-valued logic?

MCL usually appears as an extension from a two-valued to a five-valued logic: the two classical truth-values of truth and falsehood are supplemented with certainty, impossibility, and variability. But why five values, rather than any other number? The cases of five-valued logics are very rare, given their uncommon cardinality; but we can mention in this respect the recent system of Arnon Avron [2008]. This logical system is a paraconsistent logic including a set of five values: necessary and consistent truth (T), necessary and consistent falsehood (F), contingent and consistent truth (t), contingent and consistent falsehood (f), and inconsistency (I). Although MCL has nothing to do with paraconsistency, it shares with Avron's system the common idea to extend the set of truth-values by combining further properties with the initial values of truth and falsehood. Just as Avron's non-classical truth-values appear as special cases of truth and falsehood, MacColl emphasized the process of *partition* into the basic sets of truth and falsehood:

> Toutes les propositions intelligibles peuvent être divisées en deux classes, les *vraies* (t) et les *fausses* (ι). Toutes les propositions intelligibles peuvent être aussi divisées en trois classes : les *certaines*

> ($\varepsilon_1, \varepsilon_2, \varepsilon_3$, etc.), les *impossibles* (η_1, η_2, η_3, etc.) et les *variables* ($\theta_1, \theta_2, \theta_3$, etc.). [MacColl 1901, 138]

A way of reformulating this valuation is to split each basic concept of truth ($x = 1$) or falsehood ($x = 0$) into four modes of being so: plainly ($x.3$), always ($x.4$), never ($x.5$), and sometimes ($x.6$). The result is a new class of eight combined elements: *Eight* ={1.3, 1.4, 1.5, 1.6, 0.3, 0.4, 0.5, 0.6}. While noting that two pairs of these elements are synonymous and reducible to each other, *i.e.* 1.4-0.5 and 0.4-1.5, MacColl's modalities correspond to the following subclasses of elements:

$\varepsilon : \{1.4\}$

$\eta : \{0.4\}$

$\tau : \{1.3, 1.4, 1.6\}$

$\iota : \{0.3, 0.4, 0.6\}$

$\theta : \{1.6, 0.6\}$

This numerical translation of the modalities helps us see that MacColl's five elements overlap each other: the basic set of truth (1) includes both plain truth (τ for $\{1.3, 1.4, 1.6\}$) and certain truth (ε for $\{1.4\}$) as its elements, just as the basic set of falsehood (0) includes plain falsehood (ι for $\{0.3, 0.4, 0.6\}$) and certain falsehood (η for $\{0.4\}$). In symbols: 1 corresponds to $\{1.3, 1.4, 1.5, 1.6\}$, and 0 to $\{0.3, 0.4, 0.5, 0.6\}$. MCL would thus refer to the following semantic class which includes five subclasses of *Eight*, each of these corresponding to a modality from the set *Five*: $\{\{1.4\}, \{0.4\}, \{1.4, 1.5, 1.6\}, \{0.4, 0.5, 0.6\}, \{1.6, 0.6\}\}$.

Does the cardinality of *Five* entail that MCL is a many-valued logic, however, in the sense that it contains more than two elements (values)? Against this view, Peter Simons argues that three preconditions have to be required for a semantic class to be "essentially many-valued" [Simons 1998, 86]. The first (MV1) is: to contain at least one other value *besides* truth and falsehood. The second (MV2) is: for all and only all the elements to be *pairwise exclusive* and *jointly exhaustive*. The third (MV3) is: for the connectives to be *value-functional*.

Simons concludes against Nicholas Rescher [Rescher 1969] that MCL is not an essentially many-valued logic, because no one of these three criteria are respected by it. On the one hand, it fails to satisfy (MV1) insofar as the three additional values are not introduced into *Five* besides truth and falsehood: ε ($\{1.4\}$) belongs to τ ($\{1.3, 1.4, 1.6\}\}$ within the truth-class 1, since whatever is true is so plainly (without qualification), necessarily (always), or contingently (sometimes, but not always); the same goes for η and ι, with respect to the falsehood-class 0. On the other hand, not all the elements of *Five* are needed to exhaust the semantic class: τ and ι are sufficient, as witnessed by the theorem (T12) $(A^\tau + A^\iota)^\varepsilon$. This is because truth and falsehood are subclasses whose joint elements exhaust all the modes of truth. And finally, Simons contends that MCL is not value-functional because the operations $\theta.\theta$ and $\theta : \theta$ don't have any determinate result [Simons 1998, 87].

Let us review each of Simons's objections to the uncritical view that MCL is a many-valued logic.

About MV1, firstly. The preceding reformulation helps to bring out some inferential relations between the five predicates. Thus, the theorems (T15) $A^{\varepsilon} : A^{\tau}$ and (T16) $A^{\eta} : A^{\iota}$ state nothing but an inclusive relation between the antecedent and its consequent: $\{1.4\} \subset \{1.4, 1.5, 1.6\}$ and $\{0, 4\} \subset \{0.4, 0.5, 0.6\}$. But this is not a sufficient condition to consider the resulting set of inferences as really different from classical logic, Simons claims. He gives an example of a four-valued matrix for negation and conjunction, where each element of the corresponding semantic class {(1),(2),(3),(4)} comes from the Cartesian product {0,1}×{0,1} of the classical truth-values 1 (for truth) and 0 (for falsehood). Thus (1) = 11, (2) = 10, (3) = 01, and (4) = 00. On the basis of this matrix, Simons rightly notes that there is no difference between the resulting set of logical theorems and that of classical logic, which means that both logics are one and the same, despite the contrary appearances of the valuations. It follows from it that MCL is not many-valued if, by a many-valued logic, we mean a deviant system that does not contain every theorem of classical logic; given that excluded middle, non-contradiction and all other classical laws are preserved, we agree with Simons and claim that MCL could be viewed rather as a modal extension of classical logic.

At any rate, the criterion (MV1) makes use of a technical device: *product systems*, *i.e.* this branch of (allegedly) many-valued logic where each logical value is the Cartesian product of classical values. Such a technique has been used by Prior [Prior 1955] to evaluate tensed propositions and is strikingly reminiscent of MacColl's modalities. Let $W = \{w, w*\}$ be a set of two different states of affairs (or truth-cases) w and $w*$. Then the same valuations as above can be produced to give an intuitive interpretation of necessity and possibility: whatever is necessary is true in *every* "possible world" w and $w*$, while whatever is possible is true in *at least one* of these. Apart from the probabilistic tone of MacColl's modalities (assuming a *finite* set W of elements that enables one to assess the ratio between true cases and false cases), the same view of modalities occurs in Prior's logic and MCL. This will be reviewed in a more detailed way in the next section.

Although the failure of (MV1) is a sufficient ground for Simons to settle the problem and claim that MCL is not many-valued, let us consider the two other points: they should throw some new light upon MacColl's modalities, whether they are many-valued or not.

About (MV2), secondly. Simons is also right to affirm that truth and falsehood are exhaustive, given that they cover all the modes of being true and false as their particular cases. Hence the validity of the theorems (T15) and (T16), again; but this result contains a further subtlety that is not mentioned by Simons: any proposition A is a theorem in MCL if and only if it is certain, *i.e.* $A = \varepsilon$. Now this certainty is not the same as the certainty that is included among the different modes of being true. To give an expression to this symbolic ambiguity, MacColl makes a difference between two ways of being certain:

> A proposition is called a *formal certainty* when it follows necessarily from our definitions, or our understood linguistic conventions, without further data; and it is called a *formal impossibility*, when it is inconsistent with our definitions or linguistic conventions. It is called a *material certainty* when it follows necessarily from special data not necessarily contained in our definitions. Similarly, it is called a *material impossibility* when it contradicts some special datum or data not contained in our definitions. [MacColl 1906, 97]

It appears from this distinction that a theorem is *formally* certain because it is a tautology, *i.e.* its denial is made inconsistent by the definitions of MCL; by contrast, a formula is materially certain because it is not tautological but yields only truth-cases on the basis of empirical data. If one introduces a symbolic change to bring out this difference between logical and non-logical (physical) necessity, e.g. ε_f for formal certainty and ε_m for material certainty, theorem (T11) should be restated as $(A + A')^{\varepsilon_f}$, and the theorem (15) as $A^{\varepsilon_m} : A^{\tau}$.

Such a precision is not gratuitous: it is essential to make sense of the way in which MacColl thought of implication. Given that $A : B$ and A^B can be interchanged, this means that B is implied by A if and only if the class A is included in the class B. But if so, how could the class of certainty be included in the class of truth (by (T15)) and at the same time include it (by (T11))? Returning to our preceding reformulation of the modalities, this entails that a further set-theoretical definition should be introduced for formal certainty: assuming that $\varepsilon = \{1.3\}$ and $\tau = \{1.3, 1.4, 1.6\}, \varepsilon_f$ should be a set that includes the set for τ. We will pursue this line in the next section.

About (MV3), thirdly. The logical matrices for MCL are functionally incomplete, according to Simons, given that some of the operations for conjunction and implication are left undetermined. The two cases mentioned by Simons are gathered from counterintuitive examples. Thus, Simon concedes that $A^{\theta} : B^{\theta}$ is certainly true whenever A and B are one and the same proposition, for the reason that any such proposition is certainly self-identical; but it is taken to be contingently true, if both propositions are different and not related by a logical or causal relation. The same is said about $A^{\theta}.B^{\theta}$, on the other hand.

Nevertheless, two replies could be given to these objections about (MV3).

On the one hand, that implication yields counterintuitive results is not specific to MacColl's implication and need not lead to a non-value-functional matrix for MCL. No such matrix is explicitly given in MacColl's writings, unfortunately; but a recursive definition of his operations should be still in order, if we assume the set-theoretical process of inclusion between classes as a general pattern for his symbolic logic.

On the other hand, the alleged indeterminacy of $A^{\theta}.B^{\theta}$ is not justified but merely conjectured by Simons, and we suspect him of confusing variability with probability in this respect: if the probability of A is between 1/2 and 1, A is probable but its conjunction with another probable proposition B could weaken the final ratio under 1/2; if so, then $A = B = p$ and $A.B = q$. But again, the indeterminacy of $p.p$ concerns probability and does not affect the conjunction rules for variability.

As a general result, (MV1) seems to be the most plausible objection against the statement that MCL is many-valued: the given list of logical theorems is not deviant from classical logic, and the set of truth-values doesn't include any new element independent from truth and falsehood. However, a difference between the bivalent set of classicists and MCL lies in the process of *partition*. Actually, such a division of the classes of truth and falsehood into a number of subclasses explains the view defended by Rescher [Rescher 1969] that MCL is both many-valued and modal: a partition augments the cardinality of the initial set beyond two elements, and the additional elements stand for different modes of truth.

Irrespective of the proper criteria for many-valuedness or modality, let us see how to streamline MCL in such a way that its various theorems could be derived from general and recursive principles.

A non-Fregean logic!

MacColl appeared as a member of the algebraic school in logic, in the sense that he attempted to algebraize logic as Boole or Schröder did before him. Another symptom of this was the fact that MacColl regarded logic as a useful instrument, rather than a universal language for correct thought. It also turns out that every many-valued logician naturally subscribes to such an relativist or goal-dependent view of logic, by contrast to the universalist line defended by Russell, Frege or the early Wittgenstein. One corollary of this duality between relativists and universalists is the famous controversy between Russell and MacColl. According to Russell, every proposition is either true or false *and cannot be anything else*, unlike the previous partition of truth and falsehood into several modes. Russell defends his point with reference to MacColl's semantic predications:

> Either of these is a propositional function; but neither is a proposition [...] Thus we shall say that *true* and *false* are alone applicable to propositions, while *certain*, *variable* and *impossible* are applicable to ambiguous forms of words and to propositional functions. [Russell 1906, 257]

Woleński [Woleński 1998] recalled that Russell reduced modalities to quantifiers, while MacColl made a symbolic difference between quantification (A^1, A^0, A^{-0}) and modality $(A^{\varepsilon}, A^{\eta}, A^{\pi})$. But whether such a reduction is advisable or not is unclear: why should a strict bivalence be defended, and doesn't it restrict the expressive power of logic? If we assume the Platonist view that propositions are abstract objects whose truth (or falsehood) means the occurrence (or failure) of a one-one correspondence with *facts*, bivalence naturally emerges from this alternative. Otherwise, Russell's position needs to be defended in practical terms of convenience: a logic with only two truth-values should be favored because of its being equally informative and more economical. But MacColl seemingly contests this point:

> What chiefly led me to this decision was the discovery that in dealing with implications of the higher degrees (*i.e.* implications of implications) a calculus of two dimensions (unity and zero) is too limited, and that for such cases we must adopt a *three*-divisional classification of our statements. [MacColl 1897, 496]

MacColl does so in order to emphasize the way in which a proposition may be true, *i.e.* certainly or variably. But this is not a sufficient reply to Russell's objection: to turn a modal proposition in MCL into a propositional function in Frege-Russell's logic makes the latter a convenient symbolism for MacColl's modalities. At the same time, this is a sufficient reply if one wants to distinguish *formal* from *material* certainty: the latter distinction cannot be rendered by Russell's formalism, so let us consider it in detail.

Many-valuedness usually relates to a process of partition into the set of "truth-values". MacColl also talks about a number of "dimensions" related to this process. There are three dimensions in MCL, *i.e.* three modes of being true or false, while only two dimensions are explored in a two-valued logic. But to call MCL a three-dimensional logic seems strange, if five semantic predicates can be predicated of a term. The reason for such a cardinality is that, according to MacColl:

> Comme *termes* et comme *facteurs*, τ et ι sont équivalents respectivement à ε et η; mais pas toujours comme *exposants*. Car comme *terme* ou *facteur* (puisque A veut dire A^τ), $\tau = \tau^\tau = \varepsilon$, et $\iota = \iota^\tau = \eta$. [MacColl 1901, 140]

Once the predicate terms are thus restricted to three irreducible elements, MacColl suggests a generalization from 3- to 3^n-dimensional logics [MacColl 1897, 509]. There are further irreducible modes for the modes of being true, including these predicates that anticipate the modern epistemic modal logic: known to be true (κ), known to be false (λ), or doubtful (μ), *i.e.* neither known to be true nor known to be false. Thus a proposition A can be known to be true certainly ($A^{\varepsilon\kappa}$), doubtfully impossible ($A^{\eta\mu}$), etc., within a larger set of $3 \times 3 = 9$ higher-order modalities combining $n = 2$ groups of alethic $(\varepsilon, \eta, \theta)$ and epistemic (κ, λ, μ) modalities. While such an increasing partition would have been accused of *psychologism* by Russell, it clearly appears that MCL does not have a fixed number of truth-values but makes room for a family of *multiple*-valued languages: it is not a many-valued logic, or a deductively closed language where new truth-values would be added to truth and falsehood; rather, it is an open language in which the set of semantic predicates includes more than two elements and can expand indefinitely the basic values of truth and falsehood.

We want now to clarify this open language as a *product system*, in order to see which sort of deductively closed logic MCL could amount to.

For this purpose, let us introduce a *non-Fregean* logic of modalities: an algebraic logic in which the values that interpret the propositional variables are not Fregean "truth-values". In other words: the following logical values won't be "truth-values" including truth and falsehood but, rather, ordered n-tuples of answers R to corresponding questions Q about the propositions. Such a device is closely related to "many-valued" systems (or, better, multiple-valued, if one takes (MV1) into account) that had been elaborated before the rise of possible-world semantics, especially by Prior [Prior 1955; 1957] or Łukasiewicz [Łukasiewicz 1920]; but it also occurs in Nuel Belnap's four-valued logic FDE, where two main questions are asked about the state of information of a computer [Belnap 1977].

Let $W = \{w_1, w_2\}$ be a set of two states of affairs. Then two questions are asked about modalities, *i.e.* the mode of truth of a proposition A, namely: $q_1 =$ "Is A true in w_1?", and $q_2 =$ "Is A true in w_2?". Assuming that each question results in a yes-no answer (with 1 for "yes" and 0 for "no"), there are two questions and two possible answers for each question; we thus obtain a set of $2^2 = 4$ ordered answers within a general framework of m^n-valued logics (where n is the cardinality of Q and m is the cardinality of R), that is: (11), (10), (01), and (00). Most of MacColl's modalities can be rendered by these new logical values: $R(\varepsilon) = (11)$, $R(\theta) = (10)$ or (01), and $R(\eta) = (00)$. The trouble is that such a valuation is both incomplete and non-value-functional: variability is given two synonymous values instead of only one; truth and falsehood cannot appear in this valuation, given that they are *singular* predications about actual states of affairs. This establishes the fact that a probabilistic understanding of MCL is not sufficient to account for the meaning of MacColl's modalities: the mode of a truth differs from its frequency.

An alternative set of questions-answers can be found in [Smessaert 2009], giving rise to an alternative semantics (see [Schang 2011]) and characterizing modalities as *generalized quantifiers*. In order to express the modes of truth and falsehood, generalized quantification proceeds by subsuming an indefinite number of truth-cases under four subclasses. The ensuing questions are: q_1 = "Is A always false?", q_2 = "Is A actually (but not always) false?", q_3 = "Is A actually (but not always) true?", and q_4 = "Is A always true?".

For every pair of propositions A and B, the various operations of MCL can be performed upon their values $R(A)$ and $R(B)$ within a Boolean algebra $(\cap,\cup,\subset,-,1,0)$. Thus:

$1 \cap y = y$ and $0 \cap y = 0$; $1 \cup y = 1$ and $0 \cup y = y$

$-(r) = 0$ if and only if $r = 1$, and $--(r) = r$

For every value $R(A) = (r_1(A), r_2(A), r_3(A), r_4(A))$ and every binary operation $\circ \in (\cap, \cup, \subset)$:

$$R(A) \circ R(B) = (r_1(A) \circ r_1(B), r_2(A) \circ r_2(B), r_3(A) \circ r_3(B), r_4(A) \circ r_4(B))$$
$$R(AB) = R(A) \cap R(B)$$
$$R(A+B) = R(A) \cup R(B)$$
$$R(A') = (-r_1(A), -r_2(A), -r_3(A), -r_4(A)).$$

Then each of MacColl's five modalities can be reformulated as follows:

$R(\varepsilon) = (0001)$, $R(\eta) = (1000)$, $R(\tau) = (0011)$, $R(\iota) = (1100)$, and $R(\theta) = (0110)$.

The above questions do not serve to characterize the probabilistic values p and q, however: these need a questioning about all the n particular cases that are not *sorts of* cases, so that $R(A) = (r_1(A), \ldots, r_n(A))$ for p and q.

These valuations also coincide with the set-theoretical definition of implication as inclusiveness. Hence the following Boolean definition (where $R(A) \neq R(B)$):

$R(A : B) = R(A) \subset R(B)$, *i.e.* $-(R(A) \cap -R(B))$ or $(-R(A) \cup R(B))$.

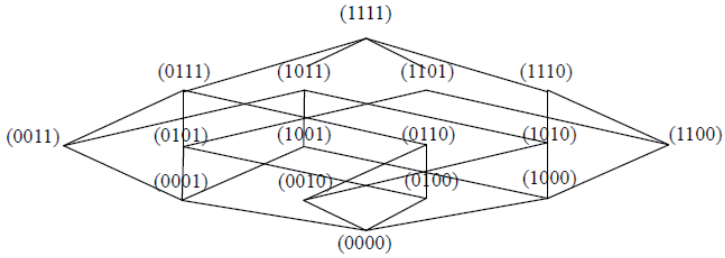
This calculus entails two further things. Firstly, implication is such that every yes-case of the antecedent is also a yes-case of the consequent, following the set-theoretical view of implication as total inclusion. That is:

$(A : B)$ holds if and only if $R(A) \cap R(B) = R(A)$ and $R(A) \cup R(B) = R(B)$.

Secondly, the difference between material and formal certainty clearly appears in the above valuation: $R(\varepsilon_m) = (0001)$, and $R(\varepsilon_f) = (1111)$. Recalling a statement of the previous section, this makes apparent that a theorem is certain in the sense of including only yes-answers in its logical value, and not in the sense of being a special mode of truth. Taking theorems (T11) and (T13) again, these respectively mean that the sum or the product of two contradictory terms A and A' is formally certain or impossible for any logical value of A: $R(A + A') = (1111)$ and $R(AA') = (0000)$ for any $R(A)$.

If this non-Fregean valuation does justice to MacColl's theorems and its view of theoremhood as formal certainty, it also means that the preceding set *Five* is largely incomplete: the number of questions and possible answers is such that there should be a total number of $m^n = 4^2 = 16$ modalities, including the five or three elements MacColl usually brought out in his writings. A presentation of these values can be made by means of the following Hasse diagram [Fig. 5], where each value is classified according to its number of yes-cases.

The lines of the Hasse diagram form implicational relations $A : B$ from bottom to top, such that the consequent B is above its antecedent A. This diagram also shows that logical truth ($\top$) is implied by everything and logical falsehood ($\bot$) implies everything, together with a number of other theorems like (T20) $\varepsilon : A = A^{\varepsilon}$, (T21) $A : \eta = A^{\eta}$, (T22) $A\varepsilon = A$ and (T23) $A\eta = \eta$.



[Fig. 5]

$(0000) = R(\tau) \cap R(\iota) = R(\eta_f) = R(\bot)$
$(0001) = R(\varepsilon)$
$(0010) = R(\tau\text{-}\varepsilon)$

$(0100) = R(\iota\text{-}\eta)$
$(1000) = R(\eta)$
$(0011) = R(\tau\text{-}\varepsilon) \cup R(\varepsilon) = R(\tau\text{-}\varepsilon + \varepsilon) = R(\tau)$
$(0101) = R(\iota\text{-}\eta) \cup R(\varepsilon) = R(\iota\text{-}\eta + \varepsilon)$
$(0110) = -(R(\eta) \cup R(\varepsilon)) = -R(\eta) \cap -R(\varepsilon) = R(\theta)$

$(1001) = -R(\theta) = R(\text{-}\theta)$
$(1010) = R(\eta) \cup R(\tau\text{-}\varepsilon) = R(\eta + \tau\text{-}\varepsilon)$
$(1100) = R(\eta) \cup R(\iota\eta) = R(\eta + \iota\text{-}\eta) = R(\iota)$

$(0111) = R(\theta) \cup R(\varepsilon) = R(\theta{+}\varepsilon) = R(\pi)$
$(1011) = R(\eta) \cup R(\tau) = R(\eta + \tau)$
$(1101) = R(\iota) \cup R(\varepsilon) = R(\iota + \varepsilon)$

$(1110) = -(R(\varepsilon)) = R(\text{-}\varepsilon) = R(u)$
$(1111) = R(\tau) \cup R(\iota) = R(\varepsilon_f) = R(\top)$

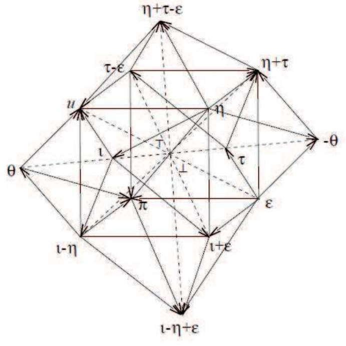
Turning again to the logical oppositions between MacColl's modalities, the corresponding logical values give rise to an algebraic characterization of the four Aristotelian oppositions. Namely:

Contrariety: A and B are contrary to each other iff $R(A) \cap R(B) = (0000)$ and $R(A) \cup R(B) \neq (1111)$
Contradiction: A and B are contradictory to each other iff $R(A) \cap R(B) = (0000)$ and $R(A) \cup R(B) = (1111)$
Subcontrariety: A and B are subcontrary to each other iff $R(A) \cap R(B) \neq (0000)$ and $R(A) \cup R(B) = (1111)$
Subalternation: B is subaltern to A iff $R(A) \cap R(B) \neq (0000)$ and $R(A) \cup R(B) \neq (1111)$ [2]

In the light of this exhaustive list of ordered answers, it clearly appears that some of the sixteen modalities are produced by meeting or joining some more basic ones: despite their symbolic appearance of simplicity, τ and ι are not basic modalities (with only one yes-case) because they equate respectively with the sum of merely actual truth and necessity or merely actual falsehood and

2. Strictly speaking, this algebraic definition of subalternation also includes the merely non-contradictory cases that do not form either a subalternation or a subcontrariety relation (see footnote 1).

impossibility. Moreover, this complete representation entails that Woleński's octagon is partial and should be completed by the tetraicosahedron of Alessio Moretti [2009] and its 120 logical oppositions [3] [Fig. 6] (only subalternations and some of the contradictory relations are drawn here, for sake of clarity):



[Fig. 6]: Pellissier's Tetraicosahedron

Contraries (18): $\{\varepsilon, \tau\text{-}\varepsilon\}$; $\{\varepsilon, \iota\text{-}\eta\}$; $\{\varepsilon, \eta\}$; $\{\varepsilon, \theta\}$; $\{\varepsilon, \iota\}$; $\{\varepsilon, \eta+\tau\text{-}\varepsilon\}$; $\{\tau\text{-}\varepsilon, \iota\text{-}\eta\}$; $\{\tau\text{-}\varepsilon, \eta\}$; $\{\tau\text{-}\varepsilon, \iota\text{-}\eta+\varepsilon\}$; $\{\tau\text{-}\varepsilon, \iota\}$; $\{\tau\text{-}\varepsilon, \text{-}\theta\}$; $\{\iota\text{-}\eta, \eta\}$; $\{\iota\text{-}\eta, \tau\}$; $\{\iota\text{-}\eta, \eta+\tau\text{-}\varepsilon\}$; $\{\iota\text{-}\eta, \text{-}\theta\}$; $\{\eta, \tau\}$; $\{\eta, \iota\text{-}\eta+\varepsilon\}$; $\{\eta, \theta\}$

Subcontraries (17): $\{\tau, u\}$; $\{\tau, \iota+\varepsilon\}$; $\{\iota\text{-}\eta+\varepsilon, u\}$; $\{\theta, \eta+\tau\}$; $\{\theta, \iota+\varepsilon\}$; $\{\iota, \pi\}$; $\{\iota, \eta+\tau\}$; $\{\eta+\tau\text{-}\varepsilon, \pi\}$; $\{\eta+\tau\text{-}\varepsilon, \iota+\varepsilon\}$; $\{\text{-}\theta, \pi\}$; $\{\text{-}\theta, u\}$; $\{\pi, \eta+\tau\}$; $\{\pi, \iota+\varepsilon\}$; $\{\pi, u\}$; $\{\eta+\tau, \iota+\varepsilon\}$; $\{\eta+\tau, u\}$; $\{\iota+\varepsilon, u\}$

Contradictories (8): $\{\varepsilon, u\}$; $\{\eta, \pi\}$; $\{\text{-}\theta, \theta\}$; $\{\tau, \iota\}$; $\{\tau\text{-}\varepsilon, \iota+\varepsilon\}$; $\{\iota\text{-}\eta+\varepsilon, \eta+\tau\text{-}\varepsilon\}$; $\{\eta+\tau; \iota\text{-}\eta\}$; $\{\top, \bot\}$

Subalterns (65): $\{\varepsilon, \tau\}$; $\{\varepsilon, \iota\text{-}\eta+\varepsilon\}$; $\{\varepsilon, \text{-}\theta\}$; $\{\varepsilon, \pi\}$; $\{\varepsilon, \eta+\tau\}$; $\{\varepsilon, \iota+\varepsilon\}$; $\{\tau\text{-}\varepsilon, \tau\}$; $\{\tau\text{-}\varepsilon, \theta\}$; $\{\tau\text{-}\varepsilon, \eta+\tau\text{-}\varepsilon\}$; $\{\tau\text{-}\varepsilon, \pi\}$; $\{\tau\text{-}\varepsilon, \eta+\tau\}$; $\{\tau\text{-}\varepsilon, u\}$; $\{\iota\text{-}\eta, \iota\text{-}\eta+\varepsilon\}$; $\{\iota\text{-}\eta, \theta\}$; $\{\iota\text{-}\eta, \iota\}$; $\{\iota\text{-}\eta, \pi\}$; $\{\iota\text{-}\eta, \iota+\varepsilon\}$; $\{\iota\text{-}\eta, u\}$; $\{\eta, \iota\}$; $\{\eta, \eta+\tau\text{-}\varepsilon\}$; $\{\eta, \text{-}\theta\}$; $\{\eta, \eta+\tau\}$; $\{\eta, \iota+\varepsilon\}$; $\{\eta, u\}$; $\{\tau, \pi\}$; $\{\tau, \eta+\tau\}$; $\{\iota\text{-}\eta+\varepsilon, \pi\}$; $\{\iota\text{-}\eta+\varepsilon, \iota+\varepsilon\}$; $\{\theta, \pi\}$; $\{\theta, u\}$; $\{\iota, \iota+\varepsilon\}$; $\{\iota, u\}$; $\{\eta+\tau\text{-}\varepsilon, \eta+\tau\}$; $\{\eta+\tau\text{-}\varepsilon, u\}$; $\{\text{-}\theta, \eta+\tau\}$; $\{\text{-}\theta, \iota+\varepsilon\}$; $\{\bot, X\}$ and $\{X, \top\}$ (for every other modality X)

Mere non-contradictories (12): $\{\tau, \iota\text{-}\eta+\varepsilon\}$; $\{\tau, \theta\}$; $\{\tau, \eta+\tau\text{-}\varepsilon\}$; $\{\tau, \text{-}\theta\}$; $\{\iota\text{-}\eta+\varepsilon, \theta\}$; $\{\iota\text{-}\eta+\varepsilon, \iota\}$; $\{\iota\text{-}\eta+\varepsilon, \text{-}\theta\}$; $\{\iota\text{-}\eta+\varepsilon, \eta+\tau\}$; $\{\theta, \iota\}$; $\{\iota, \eta+\tau\text{-}\varepsilon\}$; $\{\iota, \text{-}\theta\}$; $\{\eta+\tau\text{-}\varepsilon, \text{-}\theta\}$

The difference between internal and external negation is also made apparent by the oppositions of non-Fregean values: by reference to Piaget

3. Note that the modalities $\top$ and $\bot$ stand in a contradiction and subalternation relation at once.

[Piaget 1949], these negations correspond respectively to the two distinct operations of reciprocity and inversion in Piaget's INRC Group. Let $R(A^X) = (r_1(A^X), r_2(A^X), r_3(A^X), r_4(A^X))$ be the general form of the logical value of a proposition A^X. Then we have the following characterizations of internal negation (A'^X) and external negation ($A^{\text{-}X}$):

Internal negation: $R(A'^X) = (r_4(A^X), r_3(A^X), r_2(A^X), r_1(A^X))$

External negation: $R(A^{\text{-}X}) = (-r_1(A^X), -r_2(A^X), -r_3(A^X), -r_4(A^X))$

Let us take the example of certainty $A^X = \varepsilon$, where $r_1(\varepsilon) = r_2(\varepsilon) = r_3(\varepsilon) = 0$ and $r_4(\varepsilon) = 1$. Then its internal negation is an impossibility: $R(A'^X) = (r_4(\varepsilon), r_3(\varepsilon), r_2(\varepsilon), r_1(\varepsilon)) = (1000) = R(\eta)$; whereas its external negation is an uncertainty: $R(A^{\text{-}X}) = (-r_1(\varepsilon), -r_2(\varepsilon), -r_3(\varepsilon), -r_4(\varepsilon)) = (1110) = R(u)$. It is worth noting that internal negation corresponds to a *contrariety*-forming operator, whereas external negation is a *contradiction*-forming operator. This logical difference will be reviewed later, since MacColl made an essential use of internal negation to validate its theorems.

However useful this non-Fregean calculus may be with its Boolean understanding of MacColl's modalities, it gives rise to problems in two other respects: certainty, and denial.

The first difficulty concerns the ambiguity of MCL about certainty: although MacColl didn't make any *symbolic* distinction between the material and formal sense of certainty, the theorems (T22) $A\varepsilon = A$ and (T23) $A\eta = \eta$ cannot be validated with the material sense if one sticks to our non-Fregean valuation. For if so, then $R(A\varepsilon) = R(A) \cap R(\varepsilon) = R(A) \cap (0001) = (0000)$ whenever $r_4(A) = 0$, so that $A\varepsilon \neq A$; and $R(A\eta) = R(A) \cap R(\eta) = R(A) \cap (1000) = (0000)$ whenever $r_1(A) = 0$, so that $A\eta \neq \eta$. Nevertheless, the theorems are preserved if ε is equated with formal certainty, and it should be the case from an intuitive standpoint: the above theorems (T22) and (T23) suggest that each product takes the lowest value of its factors, and each sum the greatest; this is in accordance to our preceding Hasse diagram, where the supremum is not a material but a formal certainty. This does not involve any cancellation of material certainty, of course, given that MacColl repeatedly enhanced the opposition between two modes of certainty. A way to reconcile our formal reading of the theorems with the occurrence of material instances is to read the former as a case of *relative* necessity. Recall that MacColl endorsed the reduction of five to three main modalities $(\varepsilon, \eta, \theta)$ whenever the modalities occur as terms of factors. It is because a *statement* is given to be certainly true *if* given as true ($\tau = \tau^\tau = \varepsilon$), certainly false or impossible *if* given as false ($\iota = \iota^\tau = \eta$), and variable *if* given as variable ($\theta = \theta^\tau = \theta$). Again, the equivalence between A and A^τ stems from the nature of statement as a truth-claim. But the resulting certainty of τ^τ has nothing to do with the probabilistic meaning of material necessity; rather, it has to do with the formal certainty for a statement to be claimed to be true when taken to be true, or the formal impossibility for a statement to be claimed to be true when taken to be false. Thus MacColl argues that:

> If at the moment the servant tells me that "Mrs. Brown is not at home" I happen to see Mrs. Brown walking away in the distance, then I *have fresh data* and form the judgment A^{ε}, which, of course, implies A^{τ}. In this case I say that "A is *certain*", because its denial A' ("Mrs. Brown is at home") would contradict my data, the evidence of my eyes. [MacColl 1906, 19]

It is not the datum that is certainly true, but the linguistic rule to the effect that I state a corresponding proposition to be true once I have evidence for its truth. This rule implicitly obtains behind MacColl's symbolic language, where no clear reference is made about when a proposition occurs as a neutral propositional function or a statement taken to be true. Our context-sensitive account of certainty also argues for a formal reading of ε, since $R(\tau^{\tau}) = R(\tau : \tau) = -R(\tau) \cup R(\tau) = -(0011) \cup (0011) = (1100) \cup (0011) = (1111)$.

The second difficulty is an ambiguity in the meaning of denial. Does MacColl make an *external* or *internal* use of negation, in his definition of implication? Departing from his contemporary symbolic logicians (including Schröder) who view implication as a total inclusion of the antecedent into the consequent, MacColl said that A implies B if and only if it is impossible for A to be affirmed and B to be denied. In symbols: $A : B = (AB')^{\eta}$. Although the denial of B should refer to the external negation, as it will be in the later version of modal logic by Lewis, MacColl makes an essential use of internal negation to establish some of his theorems. An example is the paradox of strict implication, according to which certainty is implied by everything. Thus for every A in MCL, $(A : \varepsilon) = (A\varepsilon')^{\eta} = (A\eta)^{\eta} = (\eta)^{\eta} = \varepsilon$. The question is: why does one have $\varepsilon' = \eta$, rather than $\varepsilon' = \text{-}\varepsilon = u$? While MacColl patently needs an internal use of denial to validate this theorem, he argues for it for vernacular reasons:

> Some persons might reason, for example, that (...) the denial of a *possibility* is not merely an *uncertainty* but an *impossibility.* A single concrete example will show that the reasoning is not correct. The statement "It will rain tomorrow" may be considered a *possibility*; but its denial "It will *not* rain tomorrow", though an *uncertainty* is not an *impossibility.* [MacColl 1906, 15]

Two objections can be made to this explanation. On the one hand, Woleński [Woleński 1998] rightly notes that MacColl here confuses possibility and variability: the statement "It will rain tomorrow" is neither impossible nor certain, hence variable (and uncertain, since $\theta : u$); its denied form "It will not rain tomorrow" is equally variable, because this statement claims something for which no present datum is available at the time of its utterance. While it is admittedly more difficult to find a natural sentence that expresses possibility without being variable, the point is that the making of a negative statement naturally leads to an internal use of negation: "2+3 equals 5" is a certain statement and "2+3 does not equal 5" is an impossible statement, in the sense that the former is a truth-claim of something given as certain and the latter is a truth-claim of something given as impossible (by mathematical definition). Therefore, the context-sensitive interpretation of MacColl's modalities implicitly favors an internal use of negation; but this does not mean that every proposition of MCL occurs as a statement rather than a propositional function without any underlying truth-claim. Rather, a fair definition of implication should be such that no special use of negation should be preferred to the other one for merely vernacular reasons.

On the other hand, we suspect MacColl of assuming a material interpretation of certainty in the following objection to implication as total inclusion:

> If the statement $A : B$ be always equivalent to the statement $A \prec B$, the equivalence must hold good when A denotes η, and B denotes ε. Now, the statement $\eta : \varepsilon$, by definition, is synonymous with $(\eta\varepsilon')^{\eta}$, which only asserts the truism that the impossibility $\eta\varepsilon'$ is an impossibility [...] But by their definition the statement $\eta \prec \varepsilon$ asserts that the class η is wholly included in the class ε(...) Thus, $\eta : \varepsilon$ is a formal certainty, whereas $\eta \prec \varepsilon$ is a formal impossibility. [MacColl 1906, 78]

Not only does MacColl's "truism" rely upon an exclusively internal use of negation such that $\varepsilon' = \eta$; but his conclusion is wrong so long as ε is read as formal certainty while denial is applied externally. For then the validity of $\eta : \varepsilon$ trivially states that $-R\eta \cup R(\varepsilon) = -(1000) \cup (1111) = (0111) \cup (1111) = (1111)$. We find here a reason to adopt a purely set-theoretical logic of modalities in terms of class inclusion: MacColl wanted to introduce another definition of implication in order to avoid a supposedly formal impossibility, but he did so by assuming internal negation and for a partial motivation, *i.e.* the vernacular use of denial upon context-sensitive statements. Rather, our non-Fregean device validates the theorems of MCL by restoring the external use of denial in the definition of implication and favoring a formal sense of certainty for the same reasons as MacColl made a privileged use of internal negation: statements are given as certainly true in a context relative to given data, whereas the material sense of certainty still holds for other theorems of MCL (as with the T-characteristic theorems (T15) $A^{\varepsilon} : A^{\tau}$ and (T16) $A^{\eta} : A^{\iota}$). It is at this price of a relative use of denial and modalities that MCL can be streamlined into a genuinely algebraic logic of classes.

Conclusion: an algebraized *logica utens*

This paper attempted to throw some new light on MacColl's modes of modalities, *i.e.* the way in which the various modes of being true (or false) proceeded in his logical writings. Let us recapitulate our six main theses.

1. MCL is not so much a logic as a symbolic language: following a distinction of Peirce's, it is not a *logica docens*, or abstract logic without implicit

assumptions, but a *logica utens* whose results rely upon a pre-theoretical use of its symbols. MacColl's logic was not so much a closed set of theorems as a list of various implications whose explanation is not given by a totally "blind" calculus but supported by linguistic considerations. The reduction from five to three modalities witnessed this pre-theoretical use of statements as context-sensitive truth-claims.

2. MCL is a *modal* logic, in the sense that it mainly deals with modes of truth. But MacColl's modalities don't proceed like quantifiers: they occur as inclusion relations, and such a modal logic cannot be formalized in the same way as modern modal logics. The difference between iterated modalities and higher degree statements witnessed this crucial difference between modal predicates and modal operators.

3. MCL is not a *many-valued* logic if, by 'many-valued', we mean a non-classical set of theorems that includes further elements beyond truth and falsehood. Rather, it proceeds by a partition *within* these two basic elements and results in an enlarged set of multiple modes of being true or false. A better name for this logic of partitioned classes could be *multiple*-valued logic, whose theorems are a modal extension of classical logic but whose non-bivalent valuation is akin to many-valuedness.

4. MCL could be viewed as a *non-Fregean* logic, where the logical values are not truth-values but ordered answers about generalized quantifiers. The result is a multiple-valued system, doing justice to the usual view that MacColl was a father of both modal and many-valued logic. It also helps: to locate the five initial modalities within a range of sixteen logical values, to introduce a Boolean calculus for the demonstration of MacColl's theorems and, finally, to take seriously the difference between formal and material certainty as an explicit difference in their corresponding values: (1111) and (0001).

5. Our non-Fregean reconstruction showed that MacColl's definition of *implication* in terms of denial and impossibility was superfluous in two respects: on the one hand, a set-theoretical definition in terms of total inclusion fills the bill of obtaining the ensuing theorems of MCL; on the other hand, the use of internal negation can be avoided and replaced by external negation under the proviso that impossibility is read formally.

6. If we are right, then MacColl's modes of modalities can be streamlined into a pure algebraic logic. Such a result should be in perfect harmony with the theoretical background of MacColl, namely the work of Boole and Schröder, as well as that of John Venn and Louis Couturat.

It is commonly said that history is written by its winners; this is equally true for the history of logic, where the victory of Russell's ideas has largely eclipsed MacColl's writings but does not annihilate their explanatory value. The same holds in modal logic, where the victory of Kripke's possible-worlds semantics has totally eclipsed the algebraic writings about modalities by Tarski, McKinsey, Thomason or Lemmon. We hope to have recalled their relevance in this paper, through the logical legacy of MacColl as a peculiar father of algebraic modal logic.

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